

# A kind of bifurcation of limit cycle from nilpotent critical point <sup>\*</sup>

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## Abstract

In this paper, an interesting and new bifurcation phenomenon that limit cycles could be bifurcated from nilpotent node (focus) by changing its stability was investigated. It is different from lowering its multiplicity in order to get limit cycles. We prove that  $n^2 + n - 1$  limit cycles could be bifurcated by this way for  $2n + 1$  degree system. Moreover, this upper bound could be reached. At last, we give two examples to show that  $N(3) = 1$  and  $N(5) = 5$ .

**Key Words:** Nilpotent critical point; Limit cycle; Bifurcation;

## 1 Introduction and Preliminary Knowledge

One of the most intriguing aspects of the dynamics of real planar polynomial vector fields is the close relationship between the center conditions and bifurcation of limit cycle. Bifurcation of limit cycle from a high-order critical point in plane is becoming more and more important, there have been many results about this problem. The Bogdanov Takens bifurcation from saddle-node point was discussed by Xiao and Zhan and De

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Maesschalck, see [Xiao D., 2007, Xiao D., 2008, De Maesschalck P, Dumortier F., 2011] and [Tang Y, Zhang W., 2004], bifurcation of limit cycle from degenerate critical point was investigated by Han, see [Han M, Yu P., 2012]. Especially, there were many results about bifurcations of limit cycles from nilpotent critical point, see [Lyapunov, A.M., 1966, Takens, 1974, Moussu, 1982, Strozyna and Zoladek, 2002, Álvarez and Gasull, 2005] and [Álvarez and Gasull, 2006, Han M, Jiang J, Zhu H., 2008, Han M, Romanovski V G., 2012, Liu-Li, 2009a, Liu-Li, 2009b, Liu-Li, 2011a, Liu-Li, 2011b].

The following planar real systems

$$\frac{dx}{dt} = y + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j = \Phi(x, y), \quad \frac{dy}{dt} = \sum_{i+j=2}^{\infty} b_{ij}x^i y^j = \Psi(x, y). \quad (1.1)$$

whose functions of right hand are analytic at the neighborhood of origin will be discussed in this paper. The linear parts of (1.1) has double zero eigenvalues but the matrix of the linearized system of (1.1) at the origin is not identically null. The origin  $O(0, 0)$  is called a nilpotent singular point.

[Liu Y.R., 1999] gave the definition of the multiplicity of the point for

$$\begin{aligned} \frac{dx}{dt} &= \sum_{k+j=0}^n a_{kj}x^k y^j = P(x, y), \\ \frac{dy}{dt} &= \sum_{k+j=0}^m b_{kj}x^k y^j = Q(x, y) \end{aligned} \quad (1.2)$$

**Definition 1.1.** Suppose  $(x_0, y_0)$  is an isolate critical point (1.2) (real or complex), if the crossing number of  $P(x, y) = 0$  and  $Q(x, y) = 0$  at  $(x_0, y_0)$  is  $N$ , then the point  $(x_0, y_0)$  is called a  $N$ -multiple singular point of (1.3),  $N$  is called the multiplicity of the point  $(x_0, y_0)$ .

From Definition 2.1 in [Liu-Li, 2011a], we have

**Proposition 1.1.** If  $\Psi(x, y(x)) = Ax^N + o(x^N)$ ,  $A \neq 0$ , then the multiplicity of the origin is a  $N$ -multiple singular point of (1.1).

A high order singular point could be broken into some low order singular point (real or complex) by a small parameters perturbation. Now, we consider the perturbed system

of (1.1) and (1.2)

$$\frac{dx}{dt} = \Phi(x, y) + h(x, y, \varepsilon), \quad \frac{dy}{dt} = \Psi(x, y) + g(x, y, \varepsilon), \quad (1.3)$$

and

$$\frac{dx}{dt} = P(x, y) + h(x, y, \varepsilon), \quad \frac{dy}{dt} = Q(x, y) + g(x, y, \varepsilon), \quad (1.4)$$

where  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l)$  is a finite dimension small parameters,  $h(x, y, \varepsilon)$  and  $g(x, y, \varepsilon)$  are power series of  $(x, y, \varepsilon)$  with nonzero convergence radius, and  $h(x, y, \mathbf{0}) = g(x, y, \mathbf{0}) = 0$ . From Theorem 1 in [Liu Y.R., 1999] and Theorem 2.1 in [Liu-Li, 2011a], it is easy to get the following theorem.

**Theorem 1.1.** *Suppose the origin of system (1.1) ( or (1.2)) is a  $N$ -multiple singular point, then when  $\|\varepsilon\| < 1$ , the sum of multiplicity of all complex singular point in the sufficiently small neighborhood of origin of (1.3) ( or (1.4)) is exactly  $N$ .*

**example 1.1.** *From Proposition 1.1, the multiplicity of the nilpotent origin of system*

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = Ax^N + yg(x, y) \quad (1.5)$$

*is exactly  $N$ ,  $A \neq 0$ ,  $g(x, y)$  is analytic in the neighborhood of origin. When  $\|\varepsilon\| \ll 1$ , there are  $m$  critical points  $(\varepsilon_k, 0)$  in the neighborhood of origin of system*

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = A \prod_{k=1}^m (x - \varepsilon_k)^{l_k} + yg(x, y), \quad (1.6)$$

*and their multiplicity are  $l_k$ ,  $k = 1, 2, \dots, m$ , where  $l_1 + l_2 + \dots + l_m = N$ .*

**Theorem 1.2.** *Suppose the index of the origin of system (1.1) ( or (1.2)) is  $k$ , then when  $\|\varepsilon\| < 1$ , the sum of index of all real singular point in the sufficient small neighborhood of origin of (1.3) ( or (1.4)) is exactly  $k$ .*

Liu etc gave the following definitions in order to compute Lyapunov constant in [Liu-Li-Huang, 2008].

**Definition 1.2.** *Let  $f_k$ ,  $g_k$  be polynomials with respect to  $a_{ij}'s, b_{ij}'s$ ,  $k = 1, 2, \dots$ . If for an integer  $m$ , there exist polynomials with respect to  $a_{ij}'s, b_{ij}'s$ :  $\xi_1^{(m)}, \xi_2^{(m)}, \dots$ ,*

$\xi_{m-1}^{(m)}$ , such that

$$f_m = g_m + \left( \xi_1^{(m)} f_1 + \xi_2^{(m)} f_2 + \cdots + \xi_{m-1}^{(m)} f_{m-1} \right). \quad (1.7)$$

Then, we say that  $f_m$  and  $g_m$  is algebraic equivalent, written by  $f_m \sim g_m$ . If for any integer  $m$ , we have  $f_m \sim g_m$ , we say that the sequences of functions  $\{f_m\}$  and  $\{g_m\}$  are algebraic equivalent, written by  $\{f_m\} \sim \{g_m\}$ .

The authors have proved that a nilpotent-node (nilpotent-focus) point with multiplicity  $2m+1$  could be broken into a nilpotent-node (nilpotent-focus) with multiplicity  $2m-1$  and two complex singular points by a small parameters perturbation in [Liu-Li, 2011a]. If the stability at the element focus and nilpotent singular point is different, limit cycle will be bifurcated out from sufficiently small neighborhood of the element focus. In this paper, bifurcation of limit cycles from a nilpotent-node (nilpotent-focus) point will be investigated by changing the stability of the nilpotent-node (nilpotent-focus) point when the multiplicity is not decreased. It is different from [Liu-Li, 2011a].

## 2 Stability and bifurcation of limit cycle at nilpotent node (focus)

Using theorem proved in [Zhifen Zhang-1985], we have

**Proposition 2.1.** *Suppose that the function  $y = y(x)$  satisfies  $\Phi(x, y(x)) = 0$ ,  $y(0) = 0$ , and*

$$\begin{aligned} \Psi(x, y(x)) &= \alpha_{2m+1} x^{2m+1} + o(x^{2m+1}), \quad \alpha_{2m+1} < 0, \\ \left( \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} \right)_{y=y(x)} &= \beta_{2n} x^{2n} + o(x^{2n}), \quad \beta_{2n} \neq 0, \end{aligned} \quad (2.1)$$

where  $n, m$  are positive integers, then the origin of (1.1) is a nilpotent-node (nilpotent-focus) point with multiplicity  $2m+1$ , and the origin to be a nilpotent-node if and only if one of the following conditions is satisfied:

$$\begin{aligned} C_1 : \quad & 2n < m, \quad \alpha_{2m+1} < 0; \\ C_2 : \quad & 2n = m, \quad \alpha_{2m+1} < 0, \quad \beta_{2n}^2 + 4(m+1)\alpha_{2m+1} \geq 0. \end{aligned} \quad (2.2)$$

Furthermore,

**Theorem 2.1.** *Suppose that the function  $y = y(x)$  satisfies  $\Phi(x, y(x)) = 0$ ,  $y(0) = 0$ , and (2.1) holds, then multiplicity of the origin of system (1.1) is  $2m + 1$ , Lyapunov constants are*

$$V_n = \beta_{2n}, \quad (2.3)$$

namely it is stable when  $\beta_{2n} < 0$  and unstable when  $\beta_{2n} > 0$ .

*Proof.* From the discussions in [Takens, 1974] and [Álvarez and Gasull, 2006], under conditions in theorem 2.1, system (1.1) could be transformed into Liénard system

$$\frac{du}{d\tau} = v, \quad \frac{dv}{d\tau} = \alpha_{2m+1}u^{2m+1} + \beta_{2n}vu^{2n}g(u) \quad (2.4)$$

by the following analytic changes

$$\begin{aligned} u &= x + \sum_{k+j=2}^{\infty} a'_{kj}x^ky^j, \\ v &= y + \sum_{k+j=2}^{\infty} b'_{kj}x^ky^j, \\ \frac{dt}{d\tau} &= 1 + \sum_{k+j=1}^{\infty} c'_{kj}x^ky^j, \end{aligned} \quad (2.5)$$

where  $g(u)$  is analytic at  $u = 0$ , and  $g(0) = 1$ . Let  $V = v^2 - \alpha_{2m+1}u^{2m+2}$ ,

$$\left. \frac{dV}{d\tau} \right|_{(2.4)} = 2\beta_{2n}v^2u^{2n}g(u), \quad (2.6)$$

So the conclusion in Theorem 2.1 holds.  $\square$

The Theorem 2.1 leads to the following theorem

**Theorem 2.2.** *Suppose that the function  $y = y(x)$  satisfies  $\Phi(x, y(x)) = 0$ ,  $y(0) = 0$ , and*

$$\begin{aligned} \Psi(x, y(x)) &= \alpha_{2m+1}x^{2m+1} + o(x^{2m+1}), \quad \alpha_{2m+1} < 0, \\ \left( \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} \right)_{y=y(x)} &= \sum_{k=1}^n \beta_{2k} \left( x^{2k} + o(x^{2k}) \right), \quad \beta_{2n} \neq 0, \end{aligned} \quad (2.7)$$

where  $n, m$  are positive integers, then there exist  $n - 1$  limit cycles in the neighborhood of origin of system (1.1) when

$$0 < |\beta_2| \ll |\beta_4| \ll \cdots |\beta_{2n}|, \quad \beta_{2k}\beta_{2k+2} < 0, \quad k = 1, 2, \dots, n-1. \quad (2.8)$$

**example 2.1.** From Theorem 2.2, when (2.8) holds, there exist  $n - 1$  limit cycles in the neighborhood of origin of system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x^{2m+1} + y \sum_{k=1}^n \beta_{2k} x^{2k}. \quad (2.9)$$

Suppose  $O$  is a nilpotent-node of system

$$\frac{dx}{dt} = y + \sum_{k+j=2}^{2n+1} a_{kj} x^k y^j, \quad \frac{dy}{dt} = \sum_{k+j=2}^{2n+1} b_{kj} x^k y^j, \quad (2.10)$$

we denote the number of limit cycles which could be bifurcated from origin of (2.10) by changing the stability of the nilpotent-node point when the multiplicity is not decreased by  $N(2n + 1)$ . It is easy to know that multiplicity of the nilpotent-node point  $O$  is no more than  $(2n + 1)^2$  from Bezout theorem and definition 1.1. Combining with 2.1, we could get

**Theorem 2.3.**

$$N(2n + 1) \leq n^2 + n - 1. \quad (2.11)$$

We will give two examples in Section 3 and Section 4 to show that the upper bound is arrival when  $n = 1, n = 2$  in (2.11), namely  $N(3) = 1, N(5) = 5$ .

### 3 $N(3)=1$

In this section, we will prove that  $N(3) = 1$ . Considering the following cubic system

$$\begin{aligned} \frac{dx}{dt} &= y + x^2 + \varepsilon^2 y^2 + \varepsilon^2 x^2 y - xy^2 + \varepsilon y^3 = X(x, y), \\ \frac{dy}{dt} &= -2xy - 2\varepsilon y^2 - 2x^3 - 2\varepsilon x^2 y - 2y^3 = Y(x, y). \end{aligned} \quad (3.1)$$

For system (3.1), a solution for  $X(x, y(x)) = 0, \quad y(0) = 0$  is

$$y = y(x) = -x^2 + x^5 + \varepsilon x^6 + \varepsilon^2 x^7 + (-2 + \varepsilon^3)x^8 + o(x^8), \quad (3.2)$$

and

$$Y(x, y(x)) = -2x^9 + o(x^9),$$

$$\left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \Big|_{y=y(x)} = 2\varepsilon x^2(1 - \varepsilon x) - 7x^4 + o(x^4), \quad (3.3)$$

From (3.3),  $\beta_2 = 2\varepsilon$ ,  $\beta_4 = -7$ ,  $\alpha_9 = -2 < 0$ , then  $\Delta = \beta_4^2 + 20\alpha_9 = 9 > 0$  when  $\varepsilon = 0$ , theorem 2.2 shows that

**Theorem 3.1.** *The origin of (3.1) is a nilpotent-node point of multiplicity 9, and there is a limit cycle in the neighborhood of origin of system (3.1) when  $0 < \varepsilon \ll 1$ .*

#### 4 $N(2) = 5$

In this section, we will prove that the upper bound could be reached when  $n = 2$ . A class of  $Z_2$  quintic system with 25-multiple nilpotent node  $O(0, 0)$

$$\frac{dx}{dt} = y + \sum_{k+j=3} a_{kj} x^k y^j + \sum_{k+j=5} a_{kj} x^k y^j = X(x, y),$$

$$\frac{dy}{dt} = \sum_{k+j=3} b_{kj} x^k y^j + \sum_{k+j=5} b_{kj} x^k y^j = Y(x, y), \quad (4.1)$$

where

$$a_{30} = 1, \quad a_{21} = 7\lambda_1, \quad a_{12} = \lambda_1 \lambda_3,$$

$$a_{03} = \frac{1}{8}(-1029\lambda_1^3 + 140\lambda_1^4 + 343\lambda_1^2\lambda_2 - 12\lambda_1^3\lambda_2 - 35\lambda_1\lambda_2^2 + \lambda_2^3$$

$$- 28\lambda_1^2\lambda_3 + 4\lambda_1\lambda_2\lambda_3 + 16\lambda_4 - 56\lambda_1^2\lambda_5 - 8\lambda_1\lambda_2\lambda_5),$$

$$a_{50} = 0, \quad a_{41} = \lambda_1\lambda_5, \quad a_{32} = \lambda_4,$$

$$a_{23} = \frac{1}{4}\lambda_1(-343\lambda_1^4 + 4\lambda_1^5 + 70\lambda_1^3\lambda_2 - 3\lambda_1^2\lambda_2^2 - 4\lambda_1^3\lambda_3$$

$$+ 28\lambda_4 - 196\lambda_1^2\lambda_5 + 8\lambda_1^3\lambda_5 + 4\lambda_1\lambda_3\lambda_5 - 4\lambda_1\lambda_5^2),$$

$$a_{05} = \frac{1}{16}(-50421\lambda_1^7 + 1960\lambda_1^8 - 16\lambda_1^9 + 19894\lambda_1^6\lambda_2 - 392\lambda_1^7\lambda_2 - 2744\lambda_1^5\lambda_2^2 + 16\lambda_1^6\lambda_2^2 + 154\lambda_1^4\lambda_2^3$$

$$+ 686\lambda_1^2\lambda_2\lambda_4 - 24\lambda_1^3\lambda_2\lambda_4 - 70\lambda_1\lambda_2^2\lambda_4 + 2\lambda_2^3\lambda_4 - 56\lambda_1^2\lambda_3\lambda_4 + 8\lambda_1\lambda_2\lambda_3\lambda_4 + 16\lambda_4^2 + 14406\lambda_1^5\lambda_5$$

$$+ 1960\lambda_1^6\lambda_5 - 64\lambda_1^7\lambda_5 - 4802\lambda_1^4\lambda_2\lambda_5 - 616\lambda_1^5\lambda_2\lambda_5 + 490\lambda_1^3\lambda_2^2\lambda_5 + 32\lambda_1^4\lambda_2^2\lambda_5 - 14\lambda_1^2\lambda_2^3\lambda_5$$

$$+ 392\lambda_1^4\lambda_3\lambda_5 + 64\lambda_1^5\lambda_3\lambda_5 - 56\lambda_1^3\lambda_2\lambda_3\lambda_5 - 112\lambda_1^2\lambda_4\lambda_5 - 16\lambda_1\lambda_2\lambda_4\lambda_5 - 64\lambda_1^5\lambda_5^2 + 112\lambda_1^3\lambda_2\lambda_5^2),$$

(4.2)

$$\begin{aligned}
a_{14} = & \frac{1}{8}\lambda_1(-7203\lambda_1^5 + 84\lambda_1^6 + 1813\lambda_1^4\lambda_2 - 4\lambda_1^5\lambda_2 - 133\lambda_1^3\lambda_2^2 + 3\lambda_1^2\lambda_2^3 - 84\lambda_1^4\lambda_3 \\
& + 4\lambda_1^3\lambda_2\lambda_3 + 8\lambda_3\lambda_4 - 1029\lambda_1^3\lambda_5 + 308\lambda_1^4\lambda_5 + 343\lambda_1^2\lambda_2\lambda_5 - 20\lambda_1^3\lambda_2\lambda_5 \\
& - 35\lambda_1\lambda_2^2\lambda_5 + \lambda_2^3\lambda_5 - 84\lambda_1^2\lambda_3\lambda_5 + 4\lambda_1\lambda_2\lambda_3\lambda_5 + 56\lambda_1^2\lambda_5^2 - 8\lambda_1\lambda_2\lambda_5^2), \\
b_{30} = & 0, \quad b_{21} = \lambda_1, \quad b_{12} = -\lambda_1(7\lambda_1 - \lambda_2), \\
b_{03} = & \frac{1}{4}\lambda_1(49\lambda_1^2 + 4\lambda_1^3 - 14\lambda_1\lambda_2 + \lambda_2^2), \\
b_{50} = & \lambda_1, \quad b_{41} = \lambda_1\lambda_2, \\
b_{32} = & \frac{1}{4}\lambda_1(-147\lambda_1^2 + 4\lambda_1^3 + 14\lambda_1\lambda_2 + \lambda_2^2 + 4\lambda_1\lambda_3 - 4\lambda_1\lambda_5), \\
b_{23} = & \frac{1}{8}\lambda_1(-343\lambda_1^3 + 196\lambda_1^4 + 147\lambda_1^2\lambda_2 - 12\lambda_1^3\lambda_2 - 21\lambda_1\lambda_2^2 + \lambda_2^3 \\
& - 84\lambda_1^2\lambda_3 + 12\lambda_1\lambda_2\lambda_3 + 8\lambda_4 + 56\lambda_1^2\lambda_5 - 16\lambda_1\lambda_2\lambda_5), \\
b_{14} = & -\frac{1}{8}\lambda_1(-7203\lambda_1^4 + 294\lambda_1^5 + 8\lambda_1^6 + 3430\lambda_1^3\lambda_2 - 84\lambda_1^4\lambda_2 - 588\lambda_1^2\lambda_2^2 + 6\lambda_1^3\lambda_2^2 \\
& + 42\lambda_1\lambda_2^3 - \lambda_2^4 - 294\lambda_1^3\lambda_3 - 16\lambda_1^4\lambda_3 + 84\lambda_1^2\lambda_2\lambda_3 - 6\lambda_1\lambda_2^2\lambda_3 \\
& + 56\lambda_1\lambda_4 - 8\lambda_2\lambda_4 + 98\lambda_1^3\lambda_5 + 24\lambda_1^4\lambda_5 - 84\lambda_1^2\lambda_2\lambda_5 + 10\lambda_1\lambda_2^2\lambda_5), \\
b_{05} = & \frac{1}{32}\lambda_1(-50421\lambda_1^5 - 6860\lambda_1^6 + 672\lambda_1^7 + 31213\lambda_1^4\lambda_2 + 2156\lambda_1^5\lambda_2 - 64\lambda_1^6\lambda_2 - 7546\lambda_1^3\lambda_2^2 \\
& - 196\lambda_1^4\lambda_2^2 + 882\lambda_1^2\lambda_2^3 + 4\lambda_1^3\lambda_2^3 - 49\lambda_1\lambda_2^4 + \lambda_2^5 - 1372\lambda_1^4\lambda_3 - 224\lambda_1^5\lambda_3 + 588\lambda_1^3\lambda_2\lambda_3 \\
& + 32\lambda_1^4\lambda_2\lambda_3 - 84\lambda_1^2\lambda_2^2\lambda_3 + 4\lambda_1\lambda_2^3\lambda_3 + 392\lambda_1^2\lambda_4 + 32\lambda_1^3\lambda_4 - 112\lambda_1\lambda_2\lambda_4 + 8\lambda_2^2\lambda_4 \\
& + 224\lambda_1^5\lambda_5 - 392\lambda_1^3\lambda_2\lambda_5 - 64\lambda_1^4\lambda_2\lambda_5 + 112\lambda_1^2\lambda_2^2\lambda_5 - 8\lambda_1\lambda_2^3\lambda_5).
\end{aligned} \tag{4.3}$$

will be investigated in this section.

Suppose that  $y = y(x)$  is the only solution of  $X(x, y(x)) = 0$ ,  $y(0) = 0$ ,  $y(x)$  and  $Y(x, y(x))$  are odd functions of  $x$  because (4.1) is  $Z_2$  equivalent, and  $\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\Big|_{y=y(x)}$  is even function of  $x$ . We have

$$\begin{aligned}
Y(x, y(x)) &= \alpha_{25}x^{25} + o(x^{25}), \\
\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\Big|_{y=y(x)} &= \sum_{k=1}^6 \beta_{2k}x^{2k} + o(x^{12}),
\end{aligned} \tag{4.4}$$



where

$$\begin{aligned}
\alpha_{25} &= -\frac{1}{16}\lambda_1^{10}(-343\lambda_1^2 + 4\lambda_1^3 + 70\lambda_1\lambda_2 - 3\lambda_2^2 - 4\lambda_1\lambda_3 + 8\lambda_1\lambda_5)^2, \\
\beta_2 &= 3 + \lambda_1, \quad \beta_4 \sim 3(56 + \lambda_2), \\
\beta_6 &\sim -\frac{3}{4}(-59 + 28\lambda_3 - 40\lambda_5), \\
\beta_8 &\sim -6(675 + \lambda_4 - 93\lambda_5), \\
\beta_{10} &\sim -\frac{27}{49}(477 + 4\lambda_5)(93 + \lambda_5), \\
\beta_{12} &\sim 972(477 + 4\lambda_5).
\end{aligned} \tag{4.5}$$

**Theorem 4.1.** *If*

$$\begin{aligned}
\lambda_1 &= -3 - \varepsilon_1, \quad \lambda_2 = -56 + \varepsilon_2, \quad \lambda_3 = \frac{1}{4}(-523 + 4\varepsilon_3 + 40\varepsilon_5), \\
\lambda_4 &= -9324 - \varepsilon_4 + 651\varepsilon_5, \quad \lambda_5 = -93 + 7\varepsilon_5,
\end{aligned} \tag{4.6}$$

then the origin of system (4.1) is a nilpotent-node point with multiplicity 25, when

$$0 < \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll \varepsilon_5 \ll 1, \tag{4.7}$$

there exist 5 limit cycles in the neighborhood of system (4.1).

*Proof.* From (4.5),  $\beta_{25} < 0$  when (4.6) and (4.6) hold, and

$$\begin{aligned}
\beta_2 &= -\varepsilon_1, \quad \beta_4 \sim 3\varepsilon_2, \quad \beta_6 \sim -21\varepsilon_3, \quad \beta_8 \sim 6\varepsilon_4, \\
\beta_{10} &\sim -405\varepsilon_5 + o(\varepsilon_5), \quad \beta_{12} \sim 102060,
\end{aligned} \tag{4.8}$$

and when  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0$ , we have

$$\Delta = \beta_{12}^2 + 52\alpha_{25} = 4198383900 > 0. \tag{4.9}$$

So the conclusion in 4.1 hold from theorem 2.2.  $\square$

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